An Ordinal Shapley Value for Economic Environments[1]

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Abstract

We propose a new solution concept to address the problem of sharing a surplus among the agents generating it. The sharing problem is formulated in the preferences-endowments space. The solution is defined in a recursive manner incorporating notions of consistency and fairness and relying on properties satisfied by the Shapley value for Transferable Utility (TU) games. We show a solution exists, and refer to it as an Ordinal Shapley value (OSV). The OSV associates with each problem an allocation as well as a matrix of concessions "measuring" the gains each agent foregoes in favor of the other agents. We analyze the structure of the concessions, and show they are unique and symmetric. Next we characterize the OSV using the notion of coalitional dividends, and furthermore show it is monotone in an agent's initial endowments and satisfies anonymity. Finally, similarly to the weighted Shapley value for TU games, we construct a weighted OSV as well.

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1 Introduction

A feature common to most economic environments is that the interaction among agents, be it through exchange, production or both, generates benefits shared among the participating individuals. The question of what interactions would or should occur, and what would be the resulting distribution of gains has been central to economic theory. It has been approached both from the normative and the positive point of view.

The normative point of view led to the analysis of existence and properties of allocations satisfying "desirable" criteria such as efficiency (Pareto optimality), fairness (envy-freeness and egalitarianism) and others. The positive point of view resulted in the analysis of outcomes generated by the interaction of the agents within given institutional structures, focusing first on competitive environments and later on the study of environments where agents possess varying degrees of market power. Game theory, cooperative and non-cooperative, has provided several important insights with respect to the normative and positive points of view. In this paper, we focus on the normative approach. We propose and analyze a new solution concept that satisfies appealing properties in economic environments.

Cooperative game theory has been especially useful in one particular class of economic environments, the one characterized by transferable utility (TU), where there exists a "numeraire" commodity that all agents value the same in terms of utility. For that class, there exist several popular notions of the distribution of gains, referred to as surplus sharing, the most well-known of which are the Core and the Shapley Value. These satisfy several desirable normative properties such as efficiency and group stability in the case of the core, and efficiency, fairness and consistency for the Shapley value.

Extending the notion of the Core to more general environments with non-transferable utility (NTU) is straightforward. However, the extension of the central concept of the Shapley Value turns out to be a much more demanding task. All known extensions describe the environment in the utility space, i.e., specifying feasible utility tuples, abstracting from the physical environment generating the tuples. A surplus sharing method is then a rule prescribing, for each environment, the utility profiles that the whole set (the grand coalition) of agents should receive.

The three known extensions of the Shapley value associate with each environment one or more TU games, and use their Shapley value to generate a surplus sharing method. To define such a method, Shapley (1969) associates with each environment a TU game, by means of a weights vector, giving the "worth" of each utility tuple. This TU game has a well-defined Shapley value. If this value is feasible for the original game, it is a utility profile associated with this environment. Aumann (1985) provides an axiomatization of this solution. Harsanyi (1959) suggests a different extension, by stressing the idea of equity. His solution contains the notion of coalitional "dividends" and each agent must end up with a payoff corresponding to the sum of his dividends. An axiomatization for this solution is provided in Hart (1985). Finally, Maschler and Owen (1989) and (1992), using a TU game associated with the grand coalition, provide an extension preserving the consistency properties of the Shapley value. Hart and Mas-Collel (1996) present a model of non-cooperative bargaining that yields the Maschler-Owen consistent value in environments with non-transferable utility.

A major shortcoming of the extensions of the Shapley Value is that the solutions are not invariant to order-preserving transformations of the agents' utilities. The notion of invariance has been addressed in the literature in two different ways. One approach considers *bargaining problems*, where the environment is given by the utility possibilities frontier for the whole set of agents and the disagreement point. A solution is then said to be ordinal, if it is invariant with respect to strictly increasing monotonic transformations of these entities. Shapley (1969) shows that there does not exist an ordinal, efficient and anonymous solution for the case of two agents, and constructs one for the three-agent case. Samet and Safra (2001), using constructions similar to O'Neill *et al.* (2001), provide a family of ordinal, efficient and anonymous solutions for bargaining problems with any number of agents greater than two. Safra and Samet (2001) provide yet another family of such solutions.

The second approach towards the ordinality issue considers the underlying physical environment generating the utility possibilities frontier. This approach better captures the basic structure of the environment since identical economic environments may lead to drastically different utility possibility frontiers (corresponding to different bargaining problems), by appropriate choices of utility functions that represent the same preferences.

In this approach the solution is defined in terms of the physical environment, i.e., in terms of allocations of commodity bundles.

To clarify the difference between the two approaches towards the analysis of ordinality, take the example of a two-agent exchange economy. Consider the representation of this economy as an NTU game (or equivalently, a bargaining game). Following Shapley (1969) there is no ordinal, efficient and anonymous solution concept for this game. However, it is clear there are several ordinal, efficient and anonymous solution concepts for the exchange economy such as the competitive equilibrium, the core and others. Therefore, an ordinal solution for the economic environment need not be an ordinal solution for the NTU game. Similarly, an ordinal NTU solution need not be ordinal if analyzed as a solution for the economic environment.

Pazner and Schmeidler (1978) provide a family of ordinal solutions given by Pareto-Efficient Egalitarian-Equivalent ($PEEE$) allocations for exchange economies. They consider the problem of allocating a bundle of goods among a set of agents. In their environment, each of the agents has the same *a priori* rights. An allocation is $PEEE$ if it is Pareto efficient and fair, in the sense that there exists a fixed commodity bundle (the same for each agent) such that each agent is indifferent between this bundle and what he gets in the allocation. Crawford (1979) and Demange (1984) propose procedures for implementing $PEEE$ allocations. McLean and Postlewaite (1989) consider pure exchange economies as well, and define an ordinal solution given by nucleolus allocations, extending the notion of a nucleolus defined for TU games in Schmeidler (1969). Nicolò and Perea (2002) also start from the physical environment, and provide ordinal solutions for the case of two agents that, under some conditions, also extend to environments with any number of agents.

Our work continues this line of research by proposing an *ordinal solution* based on the physical environment. This new solution incorporates several of the principles underlying the Shapley value in TU environments, and will be referred to as an *Ordinal Shapley Value* (OSV). It generalizes the fairness notion (of $PEEE$) by considering possibly different *a priori* rights (i.e., different initial endowments), and also the options agents have in any possible subgroup, and not just their own initial endowments. It is consistent in the sense that agents' payoffs are based on what they would get according to this rule when applied to sub-environments. In addition to these important properties of equity and consistency, the solution is efficient, monotonic, anonymous, and satisfies individual rationality. Also, the OSV is characterized through the use of "coalitional dividends" similar to the characterization of the Shapley value by the use of Harsanyi dividends (Harsanyi, 1959).

In the next Section we start by reviewing the Shapley value in TU environments. We characterize the Shapley value by the behavior of value differences (the change in a player's value when moving from a game with $n-1$ agents to a game with n agents), and recall the coalitional dividends approach as well. In Section 3 we describe the pure exchange economy underlying the NTU environment and introduce the OSV, building on the characterization of the Shapley value for TU environments provided in the previous section. In Section 4 we analyze the OSV for two-agent economies, and compare it to existing constructions. In Section 5, we prove that an OSV exists for any number of agents and furthermore it is individually rational. In Section 6, we start by proving the construction of the OSV satisfies a symmetry property. We then proceed to characterize the OSV via coalitional dividends, and provide further properties of the solution. In Section 7, we show how to generate a family of weighted OSVs, providing an ordinal analogue to the weighted Shapley values for TU environments. In Section 8, we conclude and discuss further directions of research.

2 The Shapley Value in *TU* environments: A New Characterization

Consider a *Transferable Utility* (TU) game (N, v), where $N = \{1, ..., n\}$ is the set of players, and $v : 2^N \to R$ is a characteristic function satisfying $v(\varnothing) = 0$, where $\varnothing$ is the empty set. For a coalition $S \subseteq N$,[1] $v(S)$ represents the total payoff that the partners in S can jointly obtain if this coalition is formed. We define a *value* as a mapping ξ which associates with every game (N, v) a vector in R^n that satisfies $\sum_{i \in N} \xi_i(N, v) = v(N)$.

[1] Throughout the paper, we use $\subseteq$ to denote the weak inclusion and $\subset$ to denote the strict inclusion.

The *Shapley value* (Shapley, 1953a) of every agent $i \in N$ in the *TU* game (N, v) is:

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)],$$

where $|S|$ denotes the cardinality of the subset S. The Shapley value can be interpreted as the expected marginal contribution made by a player to the value of a coalition, where the distribution of coalitions is such that any ordering of players is equally likely.

The next theorem provides a new characterization of the Shapley value, the interpretation of which follows the theorem.

Theorem 1 *A value ξ is the Shapley value if and only if it satisfies:*

$$\sum_{i \in N \setminus j} (\xi_i(N, v) - \xi_i(N \setminus j, v)) = \sum_{i \in N \setminus j} (\xi_j(N, v) - \xi_j(N \setminus i, v)) \tag{1}$$

for all $j \in N$ and for all (N, v).

Proof. To prove that the Shapley value satisfies the equality note that (1) is equivalent (rearranging terms and using $\sum_{i \in N} \xi_i(N, v) = v(N)$) to:

$$\xi_j(N, v) = \frac{1}{n} [v(N) - v(N \setminus j)] + \frac{1}{n} \sum_{i \in N \setminus j} \xi_j(N \setminus i, v). \tag{2}$$

It is easy to check that the Shapley value satisfies (2). (This equality has been previously used by Maschler and Owen (1989) and Hart and Mas-Colell (1989).)

Furthermore suppose that equality (1), equivalently (2), is satisfied by the value ξ, for all $j \in N$ and for all (N, v). Since (2) provides a unique recursive way of calculating ξ starting with $\xi_i(\{i\}, v) = v(\{i\})$, it characterizes the Shapley value, which completes the proof. ■

The expression $\phi_i(N, v) - \phi_i(N \setminus j, v)$ is usually referred to as the contribution of player j to the Shapley value of player i. It corresponds to the amount that makes player i indifferent between receiving the value suggested to him in the game (N, v), or receiving this payment and reapplying the value concept to the game without player j. Theorem 1 states that a value is the Shapley value if and only if, for any player j, the sum of the contributions of player j to the other players is equal to the sum of the contributions of the other players to player j.

We refer to the difference $\phi_i(N,v) - \phi_i(N\backslash j, v)$ as a *concession*, what player j concedes to player i, and denote it by c_i^j.[2] In fact, an immediate corollary of Theorem 1 is the following:

Corollary 1 *A value ξ is the Shapley value if and only if for each game (N,v) there exists a matrix of concessions $c(N,v) \equiv (c_j^i(N,v))_{i,j\in N, i\neq j}$, with $c_j^i(N,v)$ in R for all $i,j \in N, i \neq j$, such that:*

(1) $\xi_i(N,v) = \xi_i(N\backslash j, v) + c_i^j(N,v)$ for all $i,j \in N, i \neq j$, and

(2) $\sum\limits_{i\in N\backslash j} c_i^j(N,v) = \sum\limits_{i\in N\backslash j} c_j^i(N,v)$ for all $j \in N$.

We can view part *(1)* in Corollary 1 as a *consistency* property of the Shapley value. When the $n-1$ players other than j consider the value offered to them by the solution concept, they contemplate what might happen if they decide to go on their own. However, the resources at their disposal should incorporate rents they could conceivably achieve by cooperating with j. We call these rents the concessions of j to the other players.

Since the value is efficient and due to the consistency property of the concessions, the sum of concessions a player makes is a measure of the surplus left to others, once he has been compensated according to the solution concept. Therefore, part *(2)* can be interpreted as a *fairness* requirement: the concessions balance out, the sum of concessions one player makes to the others equals the sum of concessions the others make to him.

We now briefly describe some characteristics of the concessions.

For a *TU* game (N,v), for any coalition $S \subseteq N$, let the game w_S be the unanimity game (i.e., $w_S(T) = 1$ if $T \supseteq S, w_S(T) = 0$ otherwise). It is well known that the characteristic function v can be written as linear combination of unanimity games: $v = \sum_{S\subseteq N} \alpha_S w_S$. Denoting $\lambda_S = \frac{\alpha_S}{|S|}$ for all $S \subseteq N$, the Shapley value can be written (see Harsanyi, 1959) as:

$$\phi_i(N,v) = \sum_{\substack{S\ni i\\ S\subseteq N}} \lambda_S \text{ for all } i \in N. \tag{3}$$

[2] See also Pérez-Castrillo and Wettstein (2001), where concessions are interpreted as bids.

It follows that:

$$c_i^j(N,v) = \sum_{\substack{S \ni i,j \\ S \subseteq N}} \lambda_S \text{ for all } i,j \in N, i \neq j.$$

An immediate implication of the previous equality is that, in TU games, the concessions are symmetric in the sense that what player j concedes to i is the same as what player i concedes to j. The symmetry of the concessions corresponds to the balanced contributions property (see Myerson, 1980).

Another interesting property of the concessions is that, although they can in general be positive of negative, they are always non-negative if the game is convex. The game (N,v) is convex if, for all $S,T \subseteq N$ with $S \subset T$ and $i \notin T$ we have:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Next proposition states the result.

Proposition 1 *If the TU game (N,v) is convex, all the concessions $c_i^j(N,v)$ are non-negative.*

Proof. The concession $c_i^j(N,v) = \phi_i(N,v) - \phi_i(N \backslash j, v)$ is the difference between the Shapley value of agent i in the game with all the agents and agent i's Shapley value in the game without agent j. Sprumont (1990) showed that for convex games the Shapley value is a population monotonic allocation scheme. Each agent's Shapley value increases as the coalition to which he belongs expands. Thus, $\phi_i(N,v) - \phi_i(N \backslash j, v) \geq 0$ and hence the concessions are non-negative. ■

To complete the section, we point out that a value can be expressed in terms of the "Harsanyi dividends" (they are also called coalitional dividends), given in equation (3) if and only if it is the Shapley value. We return to this characterization when analyzing the properties of our proposal.

Proposition 2 *A value ξ is the Shapley value if and only if, for any game (N,v) there exists $\mu_S \in R$ for all $S \subseteq N$ such that,*

$$\xi_i(T,v) = \sum_{\substack{S \ni i \\ S \subseteq T}} \mu_S \text{ for all } i \in T, \text{ for all } T \subseteq N. \tag{4}$$

Proof. The fact that the Shapley value satisfies this property was shown by Harsanyi (1959) and it is stated in (3). To show the sufficiency we note that (4) implies that ξ is an egalitarian solution and hence must be the Shapley value (see Mas-Colell, Whinston and Green (1995, pp. 680-681) for the definition of an egalitarian solution and the fact it coincides with the Shapley value). ■

In the next section we describe the NTU environment, and define an ordinal solution concept. As will be evident from the construction it generalizes the Shapley value notion, and hence we call it an Ordinal Shapley value.

3 The Environment and the Solution

We consider a pure exchange economy with a set $N = \{1, 2, ..., n\}$ of agents and $k \geq 2$ commodities. Agent $i \in N$ is described by $\{\succeq^i, w^i\}$, where $w^i \in R^k_+$ is the vector of initial endowments and $\succeq^i$ is the preference relation defined over R^k. We denote by $\succ^i$ and $\sim^i$ the strict preference and indifference relationships associated with $\succeq^i$. For each $i \in N$, the preference relation $\succeq^i$ is assumed to be continuous and strictly increasing on R^k. We let u^i be a utility function representing the preferences of agent i.

We let $w \equiv \sum\limits_{i \in N} w^i$. The set of feasible utility profiles in R^n is denoted by A and defined by:

$$A = \left\{ u \in R^n | \exists \left(x^i\right)_{i=1,...,n} \in R^{kn}, \text{ such that } u^i(x^i) = u^i,\ i = 1, ..., n \text{ and } \sum_{i \in N} x^i \leq w \right\}.$$

Agents can conceivably be better off by reallocating their initial endowments. However, it should not be possible for the utility of one agent to grow arbitrarily large if the utilities of the other agents are bounded from below. To capture this idea, we assume that, for any $u \in A$ and $i \in N$, the set $A_i(u) \equiv \{\overline{u} \in A | \overline{u}_{-i} = u_{-i}\}$ is bounded from above.[3] In this paper, any pure exchange economy that satisfies the previous requirements is referred to as an *economic environment*.

[3]For a vector $x \in R^n$ and $i \in N$, $x_{-i} \equiv (x_1,...,x_{i-1},x_{i+1},...,x_n\}$.

3.1 An Ordinal Solution

We propose a solution concept, called the *Ordinal Shapley Value* (OSV), for pure exchange economies, the construction of which relies on the notion of concessions. However, since these economies constitute NTU environments, which are described in terms of the underlying physical structure, concessions cannot be in the form of utility transfers. Concessions are expressed in terms of commodities. We measure them in terms of a "base bundle" which we take to be $e = (1, ..., 1) \in R^k$. The main characteristic of the concept proposed is that it is *ordinal*. That is, the solution associates with each economy a set of allocations that does not depend on the numerical representation of the underlying preferences of the agents. Moreover, the solution proposed is *efficient* and satisfies *consistency* and *fairness* requirements.

Any (efficient) allocation can be viewed as a sharing of the surplus generated by the possibility of exchange among the agents. What is a "fair" and "consistent" sharing? Let us first discuss the rationale of our proposal in the case of two agents. According to our proposal, a sharing is fair if the gains from cooperation are equally distributed among the two agents. A crucial question is how to measure these gains. In our proposal, the benefits from cooperation are measured in terms of e. The gain of each agent is the amount of e units that when added to his initial endowment, yields a bundle indifferent to the bundle received by the sharing. This amount of e assumes the role of the difference in values (in the TU case).

A sharing is consistent if each agent is indifferent between the sharing outcome and what he could get if he were to walk away and keep what remains of the aggregate endowment, after compensating the other agent according to the solution concept. How should the other agent be compensated or what part of the surplus can the agent who walks away, keep? We measure the surplus he can keep by the maximal amount of e units for which, when he receives a bundle indifferent to his initial endowment augmented by that amount of e units, the other agent is left with a bundle equivalent to the bundle he received in the sharing. To state these properties more succinctly we use the notion of a concession just as in the TU case. An efficient sharing is fair and consistent if there exists a pair of concessions such that the concession made by agent i to agent j

equals the concession made by agent j to agent i, and each agent is indifferent between keeping this allocation or taking the concession proposed by the other (to add to his initial endowment).

Extending this notion to the n-person case, a solution is an efficient allocation for which there exists a matrix of concessions, one from each agent to any other agent, satisfying consistency and fairness. The consistency property now requires that any set of $(n-1)$ agents should be indifferent between keeping their allocation or taking the concessions made by the remaining agent and reapplying the solution concept to the $(n-1)$-agent economy. The sum of concessions an agent makes is a measure of the surplus left to others in terms of the commodity bundle e, once he has been compensated according to the solution concept. Similar to the Hart and Mas-Collel (1989) definition of consistency, any $(n-1)$ agents are indifferent between what the solution offers them, in the n-agent economy, and what the solution prescribes if they walk away with the surplus generated in the n-agent economy, after the remaining agent receives what the original solution gave him. The recursive nature of the definition implies that this consistency property extends to coalitions of any size.

Moreover, to ensure that the allocation reached is "fair", we require the concessions to balance out, in the sense that the sum of concessions one player makes to the others equals the sum of concessions the others make to him. In other words, the surplus generated for any set of $n-1$ agents is the same as the surplus they are willing to concede to the remaining agent.

The formal definition of this solution concept, the OSV, is as follows:

Definition 1 *The Ordinal Shapley Value is defined recursively.*

$(n=1)$ In the case of an economy with one agent with preferences $\succeq^1$ and initial endowments $a^1 \in R^k$, the OSV is given by the initial endowment: $OSV(\succeq^1, a^1) = \{a^1\}$.

Suppose that the solution has been defined for any economy with $(n-1)$ or less agents.

(n) In the case of an economy $(\succeq^i, a^i)_{i\in N}$ with a set N of n agents, the $OSV\left((\succeq^i, a^i)_{i\in N}\right)$ is the set of efficient allocations $(x^i)_{i\in N}$ for which there exists an $n-$tuple of concession vectors $(c^i)_{i\in N}$ that satisfy

$n.1)$ for all $j \in N$, there exists $y(j) \in OSV\left((\succeq^i, a^i + c_i^j e)_{i\in N\setminus j}\right)$ such that $x^i \sim^i y(j)^i$

for all $i \in N \backslash j$, *and*

n.2) $\sum_{i \in N \backslash j} c_i^j = \sum_{i \in N \backslash j} c_j^i$ *for all* $j \in N$.

It is not clear *a priori* whether or not the requirements of the definition of an OSV are mutually compatible. Before proving the existence of an OSV for any economic environment, we consider in the next section the existence and properties of the OSV for economies with two agents.

It should be noted that the choice of the bundle e to measure the surplus that accrues to each agent is arbitrary. An OSV could be constructed by using any other vector in R_+^k. The following analysis is valid regardless of the particular reference bundle chosen.

Note also that this solution concept reduces to the Shapley value in economic environments that can be described as a TU environment. In such environments there is a common unit of account which can be thought of as money, and agents' preferences are (normalized) quasi linear of the form $m + u^i(x)$ where m is "money", u^i is a utility function, and x is a commodity vector. If we measure concessions in terms of money (m), our solution yields the Shapley value.

4 The solution in the two-agent economy

For a two-agent economy, an OSV is an efficient allocation for which there exists an identical concession for each agent, such that any agent is indifferent between the bundle offered to him in the allocation or taking the concession and staying on his own.

In order to characterize a solution $(x^i)_{i=1,2}$ in the two-agent economy, notice first that, by efficiency, the bundle of player 1, x^1, must be the best for him among all the allocations that leave agent 2 indifferent or better off than the bundle x^2. Moreover, agent 2 is indifferent between x^2 and $w^2 + c^1 e$, and similarly, agent 1 is indifferent between x^1 and $w^1 + c^2 e$. Given that the concessions are the same, $c \equiv c^1 = c^2$, they must satisfy the following equality:

$$
\begin{aligned}
u^1(w^1 + ce) = max_{(z^1,z^2)}\ & u^1(z^1) \\
\text{s.t. } & u^2(z^2) \geq u^2(w^2 + ce) \\
& z^1 + z^2 \leq w^1 + w^2.
\end{aligned}
$$

The solution to this equation is given by the maximal real number c (which is non-negative) that satisfies:

$$(u^1(w^1 + ce), u^2(w^2 + ce)) \in A.$$

Since preferences are strictly increasing and the sets $A_i(u)$ are bounded, the previous c exists and is *unique*. Note that the concession in the OSV depends on the initial endowments. The OSV for the two-agent economy consists of the efficient allocations (x^1, x^2) such that $u^1(x^1) = u^1(w^1 + ce)$ and $u^2(x^2) = u^2(w^2 + ce)$. When preferences are strictly quasiconcave, the OSV allocation is unique.

For the two-agent economy the OSV has a very natural graphical representation. Figure 1 depicts the OSV when $n = 2$ and there are two commodities.

[Insert Figure 1]

For two-agent economies, our proposal bears many similarities to two previous solution concepts. First, it is similar to the Pareto-Efficient Egalitarian-Equivalent ($PEEE$) allocation proposed by Pazner and Schmeidler (1978), when addressing the issue of allocating a bundle of goods among a set of agents. The OSV allocation when the two agents have the same initial endowments is a $PEEE$ allocation as well. Note that by choosing different commodity bundles to concede with, we can generate a family of OSV allocations, all of which are $PEEE$.

Nicolò and Perea (2002) also propose an ordinal solution concept for two-person bargaining situations. Their construction yields the OSV for the class of exchange economies where aggregate endowments of all the commodities are equal and are shared equally among the two agents. Furthermore, while we require indifference with respect to adding to the two agents initial endowments, multiples of e, they require indifference with respect to adding to each agent's initial endowment a multiple of the other agent's initial endowment.

5 Existence of the OSV in the general case

As noted before, it is not obvious there exists an efficient allocation for which one can find concessions satisfying the requirements imposed by the definition of an OSV. To show such allocations exist, we invoke in Theorem 2 a fixed point argument. Furthermore we show that allocations in the OSV satisfy the desirable property of individual rationality, that is, if $x \in OSV\left((\succeq^i, w^i)_{i\in N}\right)$, then $x^i \succeq w^i$, for all $i \in N$.

We first prove the following lemma which plays an important role in the proof of Theorem 2 and is used in several propositions and comments throughout the paper.

Lemma 1 *For any $u \in R^n$ in the range of the utility functions, there exists a unique vector $a \in R^n$ such that an OSV for the $n-$agent economy $(\succeq^i, w^i + a_i e)_{i\in N}$ yields the utility tuple u.*

Proof. Lemma 1 is true for $n = 1$ by monotonicity and continuity of the preferences. We assume it holds for $n-1$ and show it also holds for n. For each $j \in N$, let $(\widehat{a}_i^j)_{i\in N\backslash j}$ be the unique vector such that the economy with $(n-1)$ agents with initial endowments $(w^i + \widehat{a}_i^j e)_{i\in N\backslash j}$ has an OSV yielding the $(n-1)-$utility tuple u_{-j}.

To prove the existence of such a vector $a \in R^n$, we propose concessions $\left(c_i^j\right)_{i,j\in N, i\neq j}$ and prove that they support an OSV yielding the utility vector u. The proposal involves the unknowns a_i, for $i \in N$, as follows:

$$c_i^j = -a_i + \widehat{a}_i^j \text{ for } i, j \in N, i \neq j.$$

The proposed concessions, in order to support an OSV, must satisfy the "fairness" condition $n.2)$:

$$\sum_{i\in N\backslash j} c_i^j = \sum_{i\in N\backslash j} c_j^i \text{ for } j \in N,$$

yielding a system of linear equations given by:

$$\sum_{i\in N\backslash j} \left(-a_i + \widehat{a}_i^j\right) = -(n-1)a_j + \sum_{i\in N\backslash j} \widehat{a}_j^i \text{ for } j \in N,$$

that is,

$$(n-1)a_j - \sum_{i\in N\backslash j} a_i = \sum_{i\in N\backslash j} \widehat{a}_j^i - \sum_{i\in N\backslash j} \widehat{a}_i^j \equiv \theta^j \text{ for } j \in N.$$

Notice that $\sum_{j\in N}\theta^j=0$. It is then easy to check that the solutions for this system are all given by:

$$a_i=\frac{1}{n}\left(\theta^i-\theta^n\right)+a_n \text{ for } i\in N,$$

where $a_n\in R$.

Denote by $\widehat{a}$ the only real number such that u is efficient for an economy where the initial endowments are $\left(w^i+\frac{1}{n}\left(\theta^i-\theta^n\right)e+\widehat{a}e\right)_{i\in N}$. The existence and unicity of such an $\widehat{a}$ is implied by the continuity and monotonicity of preferences. Let us denote by $\widehat{x}\in R^{nk}$ a Pareto efficient allocation in the economy where initial endowments are given by $\left(w^i+\frac{1}{n}\left(\theta^i-\theta^n\right)e+\widehat{a}e\right)_{i\in N}$ yielding the utility profile u.

We now prove that the allocation $\widehat{x}$, is in $OSV((\succeq^i,w^i+a_ie)_{i\in N})$, with $a_i=\frac{1}{n}\left(\theta^i-\theta^n\right)+\widehat{a}$, and the concessions $c_i^j=-a_i+\widehat{a}_i^j$ for $i,j\in N$, $i\neq j$ supporting it.

First, take any set of $(n-1)$ agents, say $N\backslash j$. An economy where these agents have initial endowments $(w^i+a_ie)_{i\in N\backslash j}$ and receive concessions $\left(c_i^j\right)_{i\in N\backslash j}$ is identical, by construction, to an economy where agents' initial endowments are $\left(w^i+\widehat{a}_i^je\right)_{i\in N\backslash j}$. Hence, there is an OSV value for this $(n-1)-$agent economy where agent i's utility is u^i, for all $i\in N\backslash j$. This corresponds to the $n.1)$ requirement in the definition of an OSV for the $n-$agent economy. Furthermore, by construction, requirement $n.2)$ is satisfied for the concessions $\left(c_i^j\right)_{i,j\in N,i\neq j}$. Finally, note that $\widehat{x}$ is efficient for the $n-$agent economy with initial endowments $(w^i+a_ie)_{i\in N,i\neq j}$ and that it generates utility levels given by u.

To complete the proof of Lemma 1, we show that if an OSV for the economy $(\succeq^i,w^i+\overline{a}_ie)_{i\in N}$ yields the utility tuple u, then $\overline{a}=a$. Denote by $(\overline{c}_i^j)_{i,j\in N,i\neq j}$ the concessions associated with this OSV. For any $j\in N$, define now the vector $\widehat{\overline{a}}^j\in R^{n-1}$ by:

$$\widehat{\overline{a}}_i^j\equiv\overline{a}_i+\overline{c}_i^j,\ \text{ for } i\in N\backslash j.$$

The economy where agents' initial endowments are $(w^i+\widehat{\overline{a}}_i^je)_{i\in N\backslash j}$ is identical, by construction, to the economy with initial endowments $(w^i+\overline{a}_ie)_{i\in N\backslash j}$ when the concessions are $(\overline{c}_i^j)_{i\in N\backslash j}$. Therefore, an OSV for the $(n-1)-$agent economy $(w^i+\widehat{\overline{a}}_i^je)_{i\in N\backslash j}$ yields the utility tuple u_{-j}. The induction argument then implies that $\widehat{\overline{a}}_i^j=\widehat{a}_i^j$ for all $i\in N\backslash j$. Moreover, this argument applies to all $j\in N$. Therefore,

$$\overline{a}_i+\overline{c}_i^j=a_i+c_i^j \text{ for all } i,j\in N,i\neq j.$$

By summing, we obtain:

$$\sum_{j\in N\backslash i}(\overline{a}_i+\overline{c}_i^j)-\sum_{j\in N\backslash i}(\overline{a}_j+\overline{c}_j^i)=\sum_{j\in N\backslash i}(a_i+c_i^j)-\sum_{j\in N\backslash i}(a_j+c_j^i) \text{ for all } i\in N.$$

By the fairness condition of both matrixes $\overline{c}$ and c,

$$(n-1)\overline{a}_i-\sum_{j\in N\backslash i}\overline{a}_j=(n-1)a_i-\sum_{j\in N\backslash i}a_j \text{ for all } i\in N,$$

hence,

$$n(\overline{a}_i-a_i)=\sum_{j\in N}\overline{a}_j-\sum_{j\in N}a_j \text{ for all } i\in N.$$

Therefore, the sign of the difference $\overline{a}_i-a_i$ is independent of i. Assume, without loss of generality, that $\overline{a}_i>a_i$ for all $i\in N$. In this case, the n agents have more resources in the economy $(w^i+\overline{a}_ie)_{i\in N}$ than in the economy $(w^i+a_ie)_{i\in N}$, in contradiction to u being efficient for both economies. ■

In the following theorem we use Lemma 1 to construct a mapping, the fixed points of which, constitute the set of utilities achieved in OSV allocations.

Theorem 2 *If agents' preferences are strictly quasiconcave, the Ordinal Shapley Value is non empty and satisfies individual rationality in economic environments.*

Proof. The proof proceeds by induction. The results hold for $n=1$ (they also hold for $n=2$, as was shown in the previous section). We assume the results hold for any economy with up to $(n-1)$ agents and prove that they hold for any economy with n agents.

We consider the economy $(\succeq^i,w^i)_{i\in N}$. We proceed to construct a continuous mapping from a suitably set of bounded utility profiles for this economy into itself. The induction assumption plays a role in showing the set of fixed points of this mapping is non empty, and the OSV $((\succeq^i,w^i)_{i\in N})$ will correspond to the set of fixed points of this mapping. We also prove that all OSV allocations are individually rational.

The set of utility profiles that constitute the domain (as well as range) of the mapping is denoted by H, and defined by:

$$H\equiv\left\{u\in R^n/u \text{ is Pareto efficient given } w, \text{ and } u^i\geq u^i(0) \text{ for } i=1,...,n\right\}.$$

In economic environments, H is a bounded set (if $n-1$ players obtain at least the utility level $u^i(0)$, there is a maximum for the utility that the remaining player can reach in any feasible allocation). Moreover, the set H is homeomorphic to the $(n-1)-$unit simplex (see, for example, Proposition 4.6.1 in Mas-Colell (1985) for a similar result). For future reference we denote by H^b the "border" of H, the set of all the utility vectors for which the ith component equals $u^i(0)$ for some i. Formally,

$$H^b \equiv \{u \in H / \ u^i = u^i(0) \text{ for some } i \in N\}.$$

For any vector $u \in A$, we look at $u_{-j} \in R^{n-1}$ for all $j \in N$. Lemma 1 provides for each u_{-j} a unique vector $a^j \in R^{n-1}$ such that an OSV for the $(n-1)-$agent economy $(\succeq^i, w^i + a_i^j e)_{i \in N/j}$ yields the utility tuple u_{-j}. We let $c_i^j(u) \equiv a_i^j$. These are the concessions that agent j "needs" to make in order for the other $n-1$ agents to achieve the utility level u_{-j}.

Using the concessions $\left(c_i^j(u)\right)_{j,i \in N, j \neq i}$ we construct n "net concessions" corresponding to u by:

$$C^i(u) \equiv \sum_{j \in N \setminus i} c_j^i(u) - \sum_{j \in N \setminus i} c_i^j(u), \text{ for all } i \in N.$$

Notice that $\sum_{i \in N} C^i(u) = 0$.

We now define a mapping from H into H. Each utility profile u in H is mapped to a utility profile $\widetilde{u} \in H$ by increasing (decreasing) the components associated with positive (negative) $C^i(u)$s, making necessary adjustments to preserve feasibility and efficiency. More precisely, we let

$$D(u) \equiv \min_{i \in N, \ C^i(u) < 0} \{u^i - u^i(0)\} \text{ if } C(u) \neq 0 \in R^n.$$
$$D(u) \equiv 0 \text{ otherwise.}$$

Note that, if u is not in H^b (that is, if u is at the "interior" of H) then $D(u) > 0$ if $C(u) \neq 0$.

Consider the following vector:

$$\overline{u}(u) \equiv u + \frac{D(u)}{\max_{i \in N}\{|C^i(u)|\} + 1} C(u) = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} + \frac{D(u)}{\max_{i \in N}\{|C^i(u)|\} + 1} \begin{pmatrix} C^1(u) \\ \vdots \\ C^n(u) \end{pmatrix}.$$

Denote by $C(u)_+ \in R^n$ the vector defined as follows: $C^i(u)_+ = C^i(u)$ if $C^i(u) > 0$, and $C^i(u)_+ = 0$ if $C^i(u) \leq 0$. Similarly, denote by $C(u)_- \in R^n$ the vector that is defined by $C^i(u)_- = C^i(u)$ if $C^i(u) < 0$, and $C^i(u)_- = 0$ if $C^i(u) \geq 0$.

If $\overline{u}(u)$ is feasible and efficient, take $\widetilde{u}(u) = \overline{u}(u)$.

If $\overline{u}(u)$ is feasible but not efficient, take

$$\widetilde{u}(u) = u + \frac{D(u)}{\max\limits_{i \in N}\{|C^i(u)|\} + 1}(C(u) - \delta C(u)_-),$$

where $\delta \in (0,1)$ is the unique real number such that $\widetilde{u}(u)$ previously defined is feasible and efficient. (The efficiency requirement implies $\delta > 0$, whereas feasibility implies $\delta < 1$.)

If $\overline{u}(u)$ is not feasible, take

$$\widetilde{u}(u) = u + \frac{D(u)}{\max\limits_{i \in N}\{|C^i(u)|\} + 1}(C(u) - \delta C(u)_+),$$

where $\delta \in (0,1)$ is the unique real number such that $\widetilde{u}(u)$ previously defined is feasible and efficient. (Here, feasibility implies $\delta > 0$, whereas efficiency implies $\delta < 1$.)

To prove that $\widetilde{u}(u) \in H$, we only need to show that $\widetilde{u}^i(u) \geq u^i(0)$ for all i. If $D(u) = 0$, this property is trivially satisfied. If $D(u) > 0$ then $C(u) \neq 0$. By the definition of $D(u)$ and $\overline{u}(u)$, it is easy to check that for $i's$ for which $C^i(u) < 0$ the decrease in coordinate i is small enough so that $\overline{u}(u)^i \geq u^i(0)$. Second, if $\overline{u}(u)^i \geq u^i(0)$, then the construction of $\widetilde{u}(u)$ makes sure that also $\widetilde{u}(u)^i \geq u^i(0)$.

Claim a: The mapping $\widetilde{u}(u)$ has a interior fixed point.

To prove the claim, notice first that the mapping $\widetilde{u}(u)$ is continuous. Indeed, the function $D(u)$ is clearly continuous. Also, $C(u)$ is continuous as soon as the "concessions" $c^i_j(u)$ are a continuous function of u. By looking at the proof of Lemma 1, we see that (by construction) the $c^i_j(u)$s are a continuous function of u. Since H is homeomorphic to an n-unit simplex, the mapping $\widetilde{u}(u)$ must have a fixed point. It now remains to show that the fixed point cannot occur on the boundary. We prove it by the way of contradiction.

Suppose by way of contradiction that the fixed point u is on the boundary, that is, $\widetilde{u}(u)^i = u^i = u^i(0)$ for some $i \in N$. Assume, without loss of generality that $u^1 = u^1(0)$. We claim that $C^1(u) > 0$. First, we prove that $\sum_{i \in N\setminus 1} c^1_i(u) > 0$. Indeed, if $\sum_{i \in N\setminus 1} c^1_i(u) \leq 0$, then after the concessions are made, player 1 obtains at least the utility $u^1(w^1) > u^1(0)$

since the aggregate endowment at the disposal of the others is lower or equal to $\sum_{i\in N\backslash 1} w^i$ and the final allocation is efficient.

Second, for u^1 to equal $u^1(0)$ it is necessarily the case that $c_1^i(u) < 0$ for all $i = 2, ..., n$. Otherwise, the initial endowment of player 1 when i concedes is at least w^1 and hence, because the OSV is individually rational for any environment with $(n-1)$ agents, his final utility can not be $u^1(0)$. Therefore, $C^1(u) > 0$ if $u^1 = u^1(0)$.

Third, since the previous reasoning holds for every i with $u^i = u^i(0)$, we also know that $D(u) > 0$ since $u^i - u^i(0) > 0$ as soon as $C^i(u) < 0$ and $C^i(u) < 0$ for at least one $i \in N$ given that $C^1(u) > 0$.

Therefore, by the construction of our mapping, the utility tuple u is mapped to a point with a strictly larger utility level for agent 1 and cannot constitute a fixed point. This proves Claim a.

Claim b: A utility tuple u is a fixed point of the function $\widetilde{u}$ if and only if there exists an allocation $x \in OSV\left((\succeq^i, w^i)_{i\in N}\right)$ such that $u(x) = u$.

To prove the claim, let u be a fixed point of the previous mapping, x the feasible allocation that yields the utility level u, and c the matrix constructed using Lemma 1 (for simplicity, we write c, C, and D instead of $c(u)$, $C(u)$, and $D(u)$). We claim that c is the matrix of concessions that support x as an OSV. Given the way we constructed c, each agent is indifferent with respect to the identity of the conceding agent. Requirement $n.1$) of the definition of an OSV is then immediately seen to hold. Also requirement $n.2$) holds since, by interiority of the fixed point, $D > 0$ if $C^j < 0$ for some $j \in N$. In an interior fixed point, $C^j = 0$ for all $j \in N$. Therefore, the concessions satisfy $\sum_{i\in N\backslash j} c_i^j = \sum_{i\in N\backslash j} c_j^i$ for all $j \in N$.

Notice also that the utility corresponding to any OSV is a fixed point of our mapping by construction. Therefore, the set of utilities generated by the OSVs coincides with the set of fixed points of the mapping $\widetilde{u}(u)$.

To complete the proof of the theorem we show that every OSV allocation is individually rational for the economy $(\succeq^i, w^i)_{i\in N}$. Assume by way of contradiction that agent i receives a bundle strictly worse than w^i in an element of $OSV\left((\succeq^i, w^i)_{i\in N}\right)$. It must then be that $\sum_{i\in N\backslash j} c_j^i > 0$, hence $\sum_{i\in N\backslash j} c_i^j > 0$ as well. This however means that there exist a $j \neq i$ for which $c_i^j > 0$. Hence if agent j concedes, agent i is in an environment with $n-1$ agents and initial endowment $w^i + c_i^j e$ which is strictly larger than w^i. By the induction assumption, the OSV for this environment would be preferred to $w^i + c_i^j e$, hence strictly preferred to w^i. This is in contradiction to the original OSV yielding an outcome worse than w^i for agent i.

This concludes the proof that the OSV exists and is individually rational. ■

Theorem 2 shows the OSV exists for any economic environment where agents' preferences are strictly quasiconcave. As we have already mentioned, the proof of the theorem uses a fixed point argument, hence it is not constructive. The proof does not provide an algorithm to calculate the OSV in a particular economy, and yields no information regarding the possible unicity of the solution in particular environments.

There is, however, much more information regarding the concessions associated with OSV allocations. First, Lemma 1 implies that the matrix associated with any OSV allocation is unique. Indeed, let $x \in OSV\left((\succeq^i, w^i)_{i\in N}\right)$ and $u^i \equiv u^i(x^i)$ for all $i \in N$. For every $j \in N$, Lemma 1 says that there exists a unique vector $c^j \in R^{n-1}$ such that an allocation in $OSV\left((\succeq^i, w^i + c_i^j e)_{i\in N\setminus j}\right)$ yields the utility tuple u_{-j}. That is, there exists a unique matrix of concessions supporting x as an OSV. Second, if we identify an allocation in the OSV, then the proof of Lemma 1 indicates how to construct the unique matrix of concessions associated with this allocation.

Finally we observe that the conditions of Theorem 2 while sufficient for existence are by no means necessary. This is evident in the following example which has also been analyzed in Hart (1985, example 5.7). Consider the economic environment with three agents $(1,2,3)$ and three commodities (x_1, x_2, x_3) where preferences for non-negative consumptions and initial endowments are given by:[4]

$$u^1(x_1^1, x_2^1, x_3^1) = x_1^1 + x_2^1 \qquad w^1 = (2,2,0)$$

[4]The utility functions, as given in Hart (1985) are defined just over the non-negative orthant. Note that in our set up the utility functions need to be defined over all of R^k. This can be accomplished in several ways without affecting the OSV outcome. One option is to let the utility function equal $-\infty$ for all points outside the non-negative orthant. Alternately (to preserve continuity) the u^i's could be redefined by:

$$u^1(x_1^1,x_2^1,x_3^1) = \min\{x_1^1,2x_1^1\} + \min\{x_2^1,2x_2^1\} + \min\{0,2x_3^1\}$$

and similarly for the other two agents.

$$u^2(x_1^2, x_2^2, x_3^2) = 0.5x_1^2 + x_3^2 \qquad w^2 = (2, 0, 2)$$
$$u^3(x_1^3, x_2^3, x_3^3) = x_2^3 + x_3^3 \qquad w^3 = (0, 2, 2)$$

The OSV outcome for this environment (it also happens to be unique) is the allocation:

$$x^1 = (4, 0.3791, 0); x^2 = (0, 0, 3.2745); x^3 = (0, 1.6209, 2.725)$$

and the concessions supporting the outcome are:

$$c_2^1 = c_1^2 = 0.129\,09; c_3^1 = c_1^3 = 0.119\,28; c_3^2 = c_2^3 = 0.112\,74.$$

The associated utility profile is $(u^1, u^2, u^3) = (4.3791, 3.2745, 4.3464)$. Note the Shapley value yields the utility profile $(4.5, 3.5, 4)$ whereas the Harsanyi value yields the utility profile $(13/3, 10/3, 13/3)$.

6 Characteristics of the OSV

By definition, the OSV allocations satisfy some fairness and consistency properties. Also, Theorem 2 shows that they are individually rational. The OSV allocations however satisfy several additional appealing properties.[5] The main result of this section provides a characterization of the OSV in terms of coalitional dividends similar to the characterization obtained for the Shapley value. The first step towards this result is to show that the fact that concessions in the previous example are symmetric is not a coincidence. The concessions supporting OSV allocations are always *symmetric* as stated in Proposition 3.

Proposition 3 *If the concession matrix c supports an OSV allocation, then $c_j^i = c_i^j$ for all $i, j \in N, i \neq j$.*

Proof. The proof proceeds by induction. It is true for any economy with $n = 2$ agents by definition (the fairness condition). We assume the property is satisfied for every economy with $n - 1$ agents and show it also holds for $(\succeq^i, w^i)_{i \in N}$. Let $x \in OSV((\succeq^i, w^i)_{i \in N})$,

[5] We stress the fact that the following properties hold independently of the agents' preferences being strictly quasiconcave or not. Strict quasiconcavity is a sufficient condition for existence of the OSV, the properties hold for every OSV allocation.

and let $(c^i_j)_{i,j\in N, i\neq j}$ and $u \in R^n$ be the concessions supporting x and the utility tuple associated with it. For any agent $i \in N$, there must exist some OSV (denoted by $y(i)$) for the $(n-1)-$agent economy $(\succeq^j, w^j + c^i_j e)_{j\in N\backslash i}$ yielding the utility profile u_{-i}. Similarly, for any agent $k \in N\backslash i$ there must exist an OSV (denoted by $y(ki)$) for the $(n-2)-$agent economy $(\succeq^j, w^j + (c^i_j + c^{ki}_j)e)_{j\in N\backslash\{i,k\}}$ yielding the utility profile $u_{-\{i,k\}}$, where $(c^{ki}_j)_{k,j\in N\backslash i, k\neq j} \in R^{(n-1)(n-2)}$ is the concession matrix supporting the OSV allocation $y(i)$. By Lemma 1 there exists a unique vector $a \in R^{n-2}$ such that an OSV for the $(n-2)-$agent economy $(\succeq^j, w^j + a^j e)_{j\in N\backslash\{i,k\}}$ yields the utility tuple $u_{-\{i,k\}}$. Hence we have $w^j + a^j e = w^j + (c^i_j + c^{ki}_j)e$ for any three distinct agents $i, j, k \in N$. By permuting the roles of i and k we obtain:

$$c^i_j + c^{ki}_j = c^k_j + c^{ik}_j \text{ for any three distinct agents } i, j, k \in N. \tag{5}$$

We will now show that $c^1_2 = c^2_1$.

By (5) we have:

$$c^1_2 + c^{31}_2 = c^3_2 + c^{13}_2,$$

$$c^3_1 + c^{23}_1 = c^2_1 + c^{32}_1,$$

$$c^2_3 + c^{12}_3 = c^1_3 + c^{21}_3.$$

By the induction assumption, concessions are symmetric for any economy with $(n-1)$ agents, hence $c^{31}_2 = c^{21}_3$, $c^{13}_2 = c^{23}_1$, and $c^{32}_1 = c^{12}_3$. Using this property and summing the three previous equations, we obtain:

$$(c^3_1 - c^1_3) + (c^1_2 - c^2_1) + (c^2_3 - c^3_2) = 0.$$

We now repeat the same argument with agent 3 replaced by agents $4, ..., n$ and get the following system of equations that the concessions must satisfy:

$$(c^3_1 - c^1_3) + (c^1_2 - c^2_1) + (c^2_3 - c^3_2) = 0,$$

$$\vdots$$

$$(c_1^n - c_n^1) + (c_2^1 - c_1^2) + (c_n^2 - c_2^n) = 0.$$

Summing it up we get:

$$\{(c_1^3 - c_3^1) + ... + (c_1^n - c_n^1)\} + (n-2)(c_2^1 - c_1^2) + \{(c_3^2 - c_2^3) + ... + (c_n^2 - c_2^n)\} = 0.$$

Using the fairness requirement $n.2)$ we get:

$$(c_2^1 - c_1^2) + (n-2)(c_2^1 - c_1^2) + (c_2^1 - c_1^2) = 0.$$

Hence, $c_2^1 = c_1^2$.

Similarly it can be shown that $c_j^i = c_i^j$ for any $i, j \in N, i \neq j$. ■

The following two propositions provide a characterization of the OSV analogous to the characterization of the Shapley value in terms of coalitional dividends.

Proposition 4 *Let $x \in OSV((\succeq^i, w^i)_{i \in N})$ and denote $u^i \equiv u^i(x^i)$ for all $i \in N$. Then, there exists a unique vector $(\lambda_S)_{S \subseteq N} \in R^{2^n}$ such that*

$$u^i\left(w^i + d_i(T)e + \sum_{\substack{S \ni i \\ S \subseteq T}} \lambda_S e\right) = u^i \text{ for all } T \subseteq N, \text{ for all } i \in T, \tag{6}$$

where $d(T) \in R^{|T|}$ is the unique vector such that an element of the set $OSV((\succeq^j, w^j + d_j(T)e)_{j \in T})$ yields the utility tuple u_T.

Proof. The proof proceeds by induction. If $N = \{i\}$, then $\lambda_{\{i\}}$ exists and is unique: $\lambda_{\{i\}} = 0$.

Suppose the result holds for any economy with at most $n-1$ agents. Let $x \in OSV((\succeq^i, w^i)_{i \in N})$ and $u^i \equiv u^i(x^i)$ for all $i \in N$. Denote by $\left(c_j^i\right)_{i,j \in N, i \neq j}$ the concessions supporting x as an OSV and, for all $j \in N$, let $y(j)$ be such that $y(j) \in OSV((\succeq^i, w^i + c_i^j e)_{i \in N \setminus j})$ and $y(j)^i \sim^i x^i$ for all $i \in N \setminus j$.

Applying the induction argument, for all $j \in N$, there exists a unique $(\lambda_S(j))_{S \subseteq N \setminus j} \in R^{2^{n-1}}$ such that:

$$u^i\left(w^i + c_i^j e + d_i(T; j)e + \sum_{\substack{S \ni i \\ S \subseteq T}} \lambda_S(j) e\right) = u^i \text{ for all } T \subseteq N \setminus j, \text{ for all } i \in T,$$

where $d(T;j) \in R^{|T|}$ is the unique vector such that an element of the set $OSV((\succeq^j, w^j + c_i^j e + d_i(T;j)e)_{j\in T})$ yields the utility tuple u_T. We first claim that $\lambda_S(j) = \lambda_S(k)$ for all $S \subseteq N\backslash\{j,k\}$. Indeed, consider the economy $(\succeq^i, w^i)_{i\in S}$ and the unique vector $d(S) \in R^{|S|}$ such that an element of $OSV((\succeq^i, w^i + d^i(S)e)_{i\in S})$ yields the utility tuple u_S. By the induction argument, there is a unique vector $(\lambda_B)_{B\subseteq S} \in R^{2^{|S|}}$ such that

$$u^i\left(w^i + d_i(T)e + \sum_{\substack{B\ni i\\ B\subseteq T}} \lambda_B e\right) = u^i \text{ for all } T \subseteq S, \text{ for all } i \in T.$$

Since the vector $d(T)$ is unique, it is immediate that $d_i(T) = c_i^j + d_i(T;j) = c_i^k + d_i(T;k)$ for all $T \subseteq S, i \in T$. And since the vector $(\lambda_B)_{B\subseteq S}$ is unique, it is also immediate that $\lambda_S = \lambda_S(j) = \lambda_S(k)$.

According to the previous claim, we can propose λ_S $(= \lambda_S(j)$ for any $j \notin S)$ for any $S \subset N$. With the vector $(\lambda_S)_{S\subset N}$, the equality $u^i\left(w^i + d_i(T)e + \sum_{\substack{S\ni i\\ S\subseteq T}} \lambda_S e\right) = u^i$ holds for all $T \subset N$ and for all $i \in T$. Moreover, the vector for which the equality happens is unique. The unique value still to be found is λ_N.

For any $i \in N$, consider the value $\lambda_N(i)$ implicitly (and uniquely) defined by:

$$u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subset N}} \lambda_S e + \lambda_N(i)e\right) = u^i.$$

We complete the proof of the Proposition if we show that $\lambda_N(i) = \lambda_N(j)$ for any $i,j \in N$. By induction, for any $i,j \in N$:

$$u^i\left(w^i + c_i^j e + \sum_{\substack{S\ni i\\ S\subseteq N\backslash j}} \lambda_S e\right) = u^i = u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subset N}} \lambda_S e + \lambda_N(i)e\right),$$

hence,

$$\lambda_N(i) = c_i^j + \sum_{\substack{S\ni i\\ S\subseteq N\backslash j}} \lambda_S - \sum_{\substack{S\ni i\\ S\subset N}} \lambda_S = c_i^j - \sum_{\substack{S\supseteq\{i,j\}\\ S\subset N}} \lambda_S.$$

Similarly,

$$\lambda_N(j) = c_j^i - \sum_{\substack{S\supseteq\{i,j\}\\ S\subset N}} \lambda_S.$$

Given the symmetry of the concessions, $c_i^j = c_j^i$, $\lambda_N(i) = \lambda_N(j)$ for all $i, j \in N$, which completes the proof. ■

Borrowing the terminology used in TU environments, we refer to the vector $(\lambda_S)_{S \subseteq N}$ as the *coalitional dividends*. Although the coalitional dividends are somewhat more complex to define in our economic environment than they are in TU environments, they reflect the same idea: if $i \in S$, then λ_S is the dividend agent i obtains because he belongs to coalition S. Indeed, given that $d(N) = 0$, the final utility agent i obtains in the OSV is:

$$u^i = u^i \left(w^i + \sum_{\substack{S \ni i \\ S \subseteq N}} \lambda_S e \right).$$

The added difficulty in our framework is how to measure the value of a coalition, since the additional utility (in terms of e) that agents in a certain coalition S obtain depends upon the level of their initial endowment. Proposition 4 shows that the proper reference to measure the increase in utility is given by the level of utility at the OSV. In TU environments, the reference point is not important since the value of the coalition does not depend on the initial endowment.

It is interesting to point out that the relationship between the coalitional dividends that exists for every OSV allocation and the concessions matrix that supports this allocation, is the same as the one that exists for the Shapley value in TU environments (that was proved in Section 2). Indeed, it is easy to see that $d(N \backslash j) = (c_i^j)_{i \in N \backslash j}$ for any $j \in N$. Therefore, applying (6) to the sets N and $N \backslash j$, we obtain:

$$u^i \left(w^i + \sum_{\substack{S \ni i \\ S \subseteq N}} \lambda_S e \right) = u^i = u^i \left(w^i + c_i^j e + \sum_{\substack{S \ni i \\ S \subseteq N \backslash j}} \lambda_S e \right) \text{ for any } i \in N \backslash j,$$

hence,

$$c_i^j = \sum_{\substack{S \ni i,j \\ S \subseteq N}} \lambda_S \text{ for all } i, j \in N, i \neq j.$$

Finally, in TU environments, it is also the case that if (6) holds for all $T \subseteq N$, then the value is necessarily the Shapley value. That is, the decomposition described in the previous proposition characterizes the Shapley value. Given that, in economic environments, the OSV can in principle be non unique, the result cannot be directly extended. However, we can state a very similar result:

Proposition 5 *Let Φ be a correspondence that associates a set of efficient allocations to every economic environment $(\succeq^i, w^i)_{i\in N}$. Suppose that it satisfies property (Q):*

(Q) For all $x \in \Phi((\succeq^i, w^i)_{i\in N})$ and $u^i \equiv u^i(x^i)$ for all $i \in N$, there exists a vector $(\mu_S)_{S\subseteq N} \in R^{2^n}$ such that

$$u^i\left(w^i + b_i(T)e + \sum_{\substack{S\ni i\\ S\subseteq T}} \mu_S e\right) = u^i \text{ for all } T \subseteq N, \text{ for all } i \in T,$$

where $b(T) \in R^{|T|}$ is a vector such that an element of the set $\Phi((\succeq^j, w^j + b_j e)_{j\in T})$ yields the utility tuple u_T.

Then, Φ is a sub-correspondence of the OSV correspondence.

Proof. The proof proceeds by induction. We prove that any correspondence Φ that satisfies property (Q) for economies with at most n agents is such that $\Phi((\succeq^j, a^j)_{j\in S}) \subseteq OSV((\succeq^j, a^j)_{j\in S})$, for every economic environment $(\succeq^j, a^j)_{j\in S}$ where $|S| \leq n$.

When $n = 1$, the proof is trivial: the efficiency of Φ implies that $x = a^i$ for all $x \in \Phi((\succeq^i, a^i))$. We assume now that the result holds for up to $n-1$ agents and show it holds for n agents.

Take $x \in \Phi((\succeq^i, w^i)_{i\in N})$ and let $(\mu_S)_{S\subseteq N} \in R^{2^n}$ be the vector associated with x. Consider the matrix $c \in R^n x R^{n-1}$ defined by:

$$c_i^j = \sum_{\substack{S\ni i,j\\ S\subseteq N}} \mu_S.$$

We claim that the matrix c supports x as an OSV. First, given that $c_i^j = c_j^i$ for all $i, j \in N, i \neq j$, condition $n.2)$ of Definition 1 is satisfied. Second, to prove condition $n.1)$, take any $j \in N$ and consider the economy $(\succeq^i, w^i + c_i^j e)_{i\in N\setminus j}$. Notice that since

$$u^i\left(w^i + \sum_{\substack{S\ni i\\ S\subseteq N}} \mu_S e\right) = u^i = u^i\left(w^i + b_i(N\setminus j)e + \sum_{\substack{S\ni i\\ S\subseteq N\setminus j}} \mu_S e\right) \text{ for all } i \in N\setminus j,$$

it happens that

$$b_i(N\backslash j) = \sum_{\substack{S\ni i,j \\ S\subseteq N}} \mu_S = c_i^j \text{ for all } i \in N\backslash j.$$

Therefore, the utility tuple u_{-j} is attainable (and efficient) in the economy $(\succeq^i, w^i + c_i^j e)_{i\in N\backslash j}$ since it is attainable (and efficient) in $(\succeq^i, w^i + b_i(N\backslash j)e)_{i\in N\backslash j}$. Denote by $y(j)$ the efficient allocation that yields u_{-j}. Since for all $T \subseteq N\backslash j$, $\left(b_i(T) - c_i^j\right)_{i\in T}$ is a vector such that an element of the set $\Phi((\succeq^j, w^j + c_i^j e + [b_j(T) - c_i^j]e)_{j\in T})$ yields the utility tuple u_{-j}, the induction hypothesis ensures that $y(j) \in OSV((\succeq^i, w^i + c_i^j e)_{i\in N\backslash j})$. This proves condition $n.1)$ and concludes the proof of the proposition. ■

Therefore, the OSV correspondence is characterized as the union of the correspondences (or as the largest correspondence) that satisfy property *(Q)*.

We conclude this section by proving further properties of the OSV. The next proposition shows that the OSV is *monotonic* in initial endowments. That is, if two agents have identical preferences and furthermore, one agent has the same or more of every commodity in his initial endowment, then that agent is better off in any OSV allocation.

Proposition 6 *Consider an economic environment $(\succeq^i, w^i)_{i\in N}$ where $\succeq^j \equiv \succeq^k$ and $w^j \geq (>) w^k$ for some $j \neq k$. Then, $x^j \succeq^j (\succ^j) x^k$ for any $x \in OSV((\succeq^i, w^i)_{i\in N})$.*

Proof. The proof proceeds by induction. Consider first the case of two agents ($n = 2$) and assume $\succeq^1 \equiv \succeq^2$. Let u represent the preferences of both agents. The unique level of utility that they achieve in the OSV allocations for this economy is:

$$\underset{c\in R_+}{Max} \left(u(w^1 + ce), u(w^2 + ce)\right) \left| \left(u(w^1 + ce), u(w^2 + ce)\right) \in A \right\} .$$

It is then immediate that $w^1 \geq w^2$ implies $x^1 \succeq^1 x^2$, for $x = OSV((\succeq^i, w^i)_{i=1,2})$. Moreover, x^1 is strictly preferred to x^2 if w^1 is strictly greater than w^2.

We assume now that the property holds for economies with up to $n - 1$ agents. We prove, by contradiction, that it also holds for economies with n agents.

Without loss of generality, suppose $\succeq^1 \equiv \succeq^2$, $w^1 \geq w^2$, and $x^1 \prec^1 x^2$ for some $x \in OSV((\succeq^i, w^i)_{i\in N})$. (For notational convenience, we do the proof for the case $w^1 \geq w^2$; the proof is similar when $w^1 > w^2$.) Using property $n.1)$ in the definition of an OSV, let $y(1) \in OSV((\succeq^i, w^i + c_i^1 e)_{i\in N\backslash 1})$ be such that $u^i(y(1)^i) = u^i(x^i)$ for all $i \in N\backslash 1$, and $y(2) \in OSV((\succeq^i, w^i + c_i^2 e)_{i\in N\backslash 2})$ be such that $u^i(y(2)^i) = u^i(x^i)$ for all $i \in N\backslash 2$.

Given that $u^i(y(1)^i) = u^i(y(2)^i)$ for all $i \in N\backslash\{1,2\}$, $u^1(y(2)^1) < u^2(y(1)^2)$, $\succeq^1 \equiv \succeq^2$, and the efficiency of the allocations $y(1)$ and $y(2)$, it must be the case that the total initial resources in the economy $(\succeq^i, w^i + c_i^1 e)_{i\in N\backslash 1}$ are larger than in the economy $(\succeq^i, w^i + c_i^2 e)_{i\in N\backslash 2}$. That is,

$$\sum_{i\in N\backslash 1} c_i^1 > \sum_{i\in N\backslash 2} c_i^2.$$

By the symmetry of the concessions, $c_2^1 = c_1^2$, $c_i^1 = c_1^i$ and $c_i^2 = c_2^i$ for all $i \in N\backslash\{1,2\}$. Therefore,

$$\sum_{i\in N\backslash\{1,2\}} c_1^i > \sum_{i\in N\backslash\{1,2\}} c_2^i.$$

Let $k \in N\backslash\{1,2\}$ be such that $c_1^k > c_2^k$, and $y(k) \in OSV((\succeq^i, w^i + c_i^k e)_{i\in N\backslash k})$ be such that $u^i(y(k)^i) = u^i(x^i)$ for all $i \in N\backslash k$. In the $(n-1)-$agent economy $((\succeq^i, w^i + c_i^k e)_{i\in N\backslash k})$, it happens that $\succeq^1 \equiv \succeq^2$ and $w^1 + c_1^k e > w^2 + c_2^k e$. By the induction hypothesis, $u^1(y(k)^1) \geq u^2(y(k)^2)$, that is, $u^1(x^1) \geq u^2(x^2)$. This is in contradiction to our original hypothesis. ■

The next property, *anonymity* of the OSV is an immediate corollary of the previous proposition.

Corollary 2 *Consider an economic environment $(\succeq^i, w^i)_{i\in N}$ where $\succeq^j \equiv \succeq^k$ and $w^j = w^k$ for some $j \neq k$. Then, $x^j \backsim^j x^k$ for any $x \in OSV((\succeq^i, w^i)_{i\in N})$. Moreover, if the preferences of agents j and k are strictly quasiconcave, then $x^j = x^k$ for any $x \in OSV((\succeq^i, w^i)_{i\in N})$.*

Proof. The first part of the Corollary is immediate after Proposition 6. The efficiency of x and the strict quasiconcavity of the common preference relation $\succeq^j$, imply that $x^j = x^k$ as soon as $u^j(x^j) = u^k(x^k)$. ■

Note that the nucleolus, an ordinal solution concept for exchange economies (McLean and Postlewaite (1989)), does not satisfy the previous anonymity property. It does satisfy however the following symmetry property: If agents j and k are identical and the allocation x is in the nucleolus, then the allocation y is also in the nucleolus, where $y^j = x^k, y^k = x^j$, and $y^i = x^i$ otherwise.

7 The weighted OSV

Shapley (1953b) extends the Shapley TU value by considering nonsymmetric divisions of the surplus. He defines the (now called) *weighted Shapley value* by stipulating an exogenously given system of weights $q \in R^n_{++}$, assigning each agent i the share $q_i / \sum_{j \in N} q_j$ of the unit in each unanimity game, and defining the value as the linear extension of this operator to the set of TU games. There exist several characterizations of the weighted Shapley value. The next proposition states, without a proof, a new characterization, similar to the one provided in Corollary 1.[6]

Proposition 7 *A value ξ is the $q-$weighted Shapley value if and only if for each game (N, v) there exists a matrix of concessions $c(N, v) \equiv (c^i_j(N, v))_{i,j \in N, i \neq j}$, with $c^i_j(N, v)$ in R for all $i, j \in N, i \neq j$, such that:*

(1) $\xi_i(N, v) = \xi_i(N \backslash j, v) + c^j_i(N, v)$ for all $i, j \in N, i \neq j$, and

(2) $\sum\limits_{i \in N \backslash j} q^j c^j_i(N, v) = \sum\limits_{i \in N \backslash j} q^i c^i_j(N, v)$ for all $j \in N$.

Following the same route we took in defining the OSV, we can define a weighted value for economic environments where the weights of the agents are taken into account. We now describe an extension of the OSV which yields the $q-$ weighted OSV $(q - wOSV)$ solution, which reduces to the $q-$ weighted Shapley value in economic environments that can be described as a TU environment. The only difference with respect to the definition of the OSV lies in the "fairness" condition $n.2)$:

Definition 2 *We define the $q-$ weighted Ordinal Shapley Value recursively.*

($n = 1$) In the case of an economy with one agent with preferences $\succeq^1$ and initial endowments $a^1 \in R^k$, the $q - wOSV$ is given by the initial endowment: $q - wOSV(\succeq^1, a^1) = \{a^1\}$.

Suppose that the solution has been defined for any economy with $(n-1)$ or less agents.

(n) In the case of an economy $(\succeq^i, a^i)_{i \in N}$ with a set N of n agents, the $q-wOSV$ $((\succeq^i, a^i)_{i \in N})$ is the set of efficient allocations $(x^i)_{i \in N}$ for which there exists an $n-$tuple of concession vectors $(c^i)_{i \in N}$ that satisfy

[6] For interpretation, see also Section 4 in Pérez-Castrillo and Wettstein (2001).

$n.1$) for all $j \in N$, there exists $y(j) \in q - wOSV\left((\succeq^i, a^i + c_i^j e)_{i \in N\backslash j}\right)$ such that $x^i \sim^i y(j)^i$ for all $i \in N\backslash j$, and

$n.2$) $\sum\limits_{i \in N\backslash j} q^j c_i^j = \sum\limits_{i \in N\backslash j} q^i c_j^i$ for all $j \in N$.

It is worthwhile to notice that the "weighted fairness" condition $n.2$), together with the "consistency" requirement $n.1$) also imply in this case that the concessions that support the $q-wOSV$ are "weighted" symmetric, in that we have $q^j c_i^j = q^i c_j^i$ for all $i, j \in N, i \neq j$. Moreover, very small changes in the proof of Theorem 2 are needed, to establish existence and individual rationality of this value, for any economic environment where agents' preferences are strictly quasiconcave, which we state as:

Theorem 3 *If agents's preferences are strictly quasiconcave, the $q-$weighted Ordinal Shapley Value is non empty and satisfies individual rationality in economic environments, for any $q \in R_{++}^n$.*

8 Conclusion

This paper addressed the problem of sharing a joint surplus among the agents creating it. We looked for a method associating with each economic environment (agents described by preferences and endowments) a set of outcomes (allocations of the aggregate endowment across the agents). We showed there exists such an (ordinal) sharing method that satisfies efficiency and suitably defined notions of consistency and fairness. These notions reduce to the usual notions of consistency and fairness satisfied by the Shapley value for TU games, hence the first reason for calling this solution concept an Ordinal Shapley Value.

The OSV provided not just an allocation but also a matrix of concessions "measuring" the gains each agent foregoes in favor of the other agents. Further analysis showed these concessions were symmetric, what agent i concedes to agent j coincides with the concession of agent j to agent i. This symmetry property reduces to the balanced contributions property of the Shapley Value for TU games. The next stage of the analysis characterized the OSV providing a fixed system of coalitional dividends, which when augmented by coalition specific transfers yields allocations equivalent to the OSV outcome for all possible coalitions. Furthermore, when it is possible to find such a system for a given allocation rule, then this allocation rule is a subset of the OSV outcome. This again has its counterpart for the Shapley value in TU games. We further showed that the OSV satisfies monotonicity in initial endowments and anonymity. Finally, we constructed a family of $q-$weighted OSV's, which are the ordinal counterparts (in our setting) to the family of $q-$weighted Shapley values for TU games.

The OSV is a natural extension of the Shapley value to general environments (NTU games). The main advantage of this extension compared to previous attempts to extend the value is the fact it is ordinal, depending only on the underlying preferences and not on their numerical representation. It is also defined in the commodity space rather in the "utility" space, whereas several previous ordinal values were defined solely on the utility space (Samet and Safra, 2001). It naturally shares most of the attractive properties of the Shapley Value, and thus offers important insights augmenting those derived from another ordinal solution concept, the ordinal nucleolus (McLean and Postlewaite, 1989). The connections between the OSV and another well-known ordinal solution concept, the competitive equilibrium outcome, remain the topic of further research.

The sharing method proposed in this paper provides a solution concept for a large class of environments. It can be used to address a variety of distributional issues in more realistic environments dispensing of the need to assume quasi-linear preferences. Problems of allocating joint costs can be handled as well without restricting the environment through the quasi-linearity in "money" assumption.

Our approach has been normative and it would be interesting to construct game forms that implement it. Pérez-Castrillo and Wettstein (2001) provide a deterministic mechanism that implements the Shapley value in pure strategy Subgame Perfect Equilibrium.

The analysis throughout the paper proceeded under the assumption of complete information. Allowing for asymmetric information and examining solution concepts in such settings remains another interesting avenue of research.

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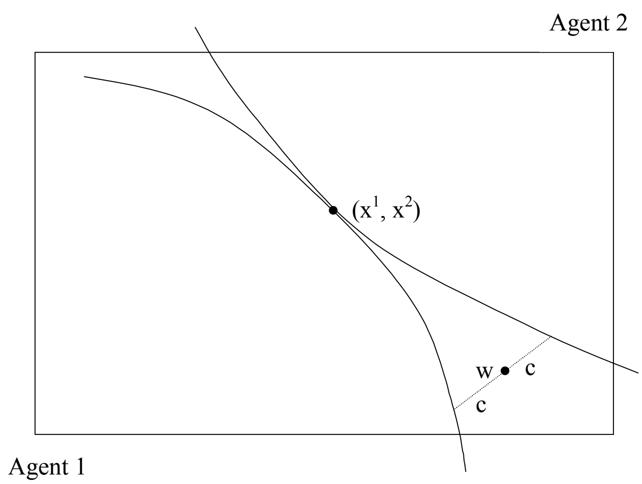

Figure 1:

The solution in the two-agent economy